~~The graphical plot of figure 8.2 should be clear: As a result of the repeated vector addition, our "polygonal spiral" gets closer and closer to the point (−1, 0) and eventually "strangles" the point. The total length of the spiral, without regard to the direction of the vectors, is simply $L = e^{\pi} = 23.1407$.~~

Which Is Larger: a^b or b^a?

In the preceding section, it was established that the total length of our "polygonal spiral" was $e^{\pi} = 23.1407$. Now, we are no longer interested in the previous topics such as complex numbers and vector diagrams. We are interested only in the quantity $e^{\pi} = 23.1407$. Suppose we switch the positions of π and e and then compute π^e. Recall that $\pi = 3.14159$ and $e = 2.71828$. With our hand calculator we determine that $\pi^e = 22.4591$, which is a bit less than $e^{\pi} = 23.1407$. Interesting, but so what?

Well, let's generalize. Suppose that a and b are positive numbers, not necessarily integers, and that a is larger than b. We ask the question: which is the larger quantity, a^b or b^a, that is, a to the bth power or b to the ath power? Incidentally, this problem was suggested by two somewhat similar problems that appear in Dörrie (1965) and Dunn (1980).

To start with, here are three numerical examples:

1. If $a = 3$ and $b = 2$, then $a^b = 3^2 = 9$ and $b^a = 2^3 = 8$. In this case, $a^b > b^a$.
2. If $a = 4$ and $b = 2$, then $a^b = 4^2 = 16$ and $b^a = 2^4 = 16$. In this case, $a^b = b^a$.
3. If $a = 5$ and $b = 2$, then $a^b = 5^2 = 25$ and $b^a = 2^5 = 32$. In this case, $a^b < b^a$.

We note that modest changes in the magnitude of a, with the value of b held constant in these three examples, entirely alter the relative magnitudes of a^b and b^a. This seems strange.

It will be very helpful in our analysis if we know when the two quantities, a^b and b^a, are equal. That is,

$$a^b = b^a. \tag{8.6}$$

Unfortunately, it is impossible to obtain a solution to this equation in a form that expresses a in terms of b or vice versa. Consequently, we must settle for a *parametric* solution. Taking the logarithms of both sides of equation (8.6) gives

$$b \log_e a = a \log_e b. \tag{8.7}$$

We let $a = kb$, where k is a number larger than one (since $a > b$); k is called the parameter. Substituting this relationship into equation (8.7) and carrying out some algebra, we obtain the expressions

$$a = k^{k/(k-1)}; \; b = k^{1/(k-1)}; \; k > 1. \tag{8.8}$$

Assigning various numerical values to the parameter k in these expressions gives the magnitudes of a and b that satisfy equation (8.6).

An interesting question is, what are the values of a and b when k approaches one? We note that the relationships of equation (8.8) are indeterminate for $k = 1$. Never mind. In these two expressions, we make the substitution $n = 1/(k - 1)$. This yields

$$a = \left(1 + \frac{1}{n}\right)^{n+1}; \; b = \left(1 + \frac{1}{n}\right)^{n}. \tag{8.9}$$

It is clear that if $k = 1$ then $n = \infty$. Utilizing equation (7.5), it is easily established that in this limiting ($k = 1$) case we have the values $a = e$ and $b = e$.

The results of our analysis are displayed in figure 8.3. For your information, table 8.1 lists some coordinates for the curve $a^b = b^a$ shown in the figure. Two interesting observations: (1) The values $a = 4$, $b = 2$ are the only *integers* for which $a^b = b^a$. (2) The values $a = e$, $b = e$, as the end point ($k = 1$) coordinates on the $a^b = b^a$ curve, represent an unexpected appearance of our very remarkable number e.

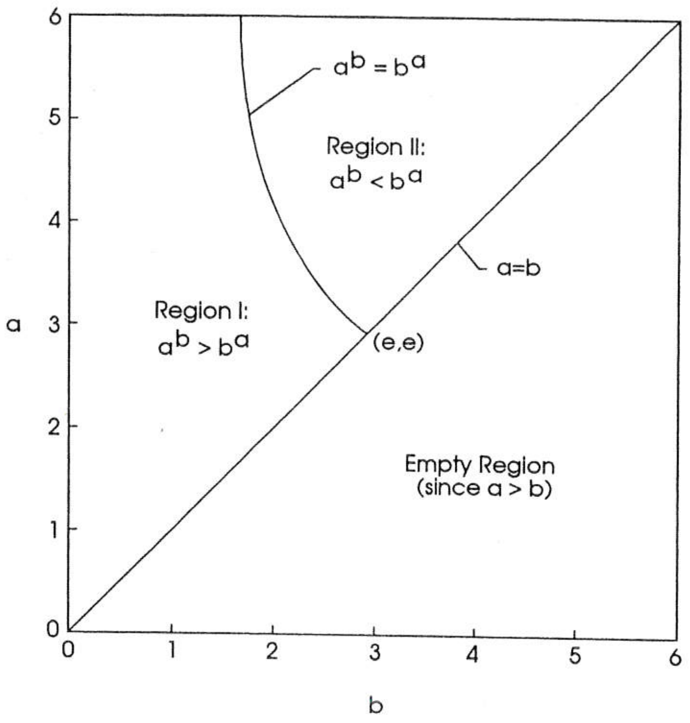

FIG. 8.3

Graphical display of the regions in which a^b is greater than, equal to, and less than b^a.

TABLE 8.1

Coordinates for the curve $a^b = b^a$ shown in figure 8.3

k	a	b
1.0		
1.1	2.853	2.594
1.2	2.986	2.488
1.6	3.502	2.189
2.0	4	2
3.0	5.196	1.732
4.0	6.349	1.587
6.0	8.586	1.431
10.0	12.915	1.292
∞	∞	1

The main feature of the display of figure 8.3 is the identification of region I in which $a^b > b^a$ and region II where $a^b < b^a$. You might want to confirm these results by computation with some selected values for a and b, including $a = \pi$ and $b = e$. Also, you will undoubtedly want to show that the slope of the $a^b = b^a$ curve at the point (e, e) is $(da/db) = -1$, i.e., it is perpendicular to the line $a = b$.